INSTITUTO POLITÉCNICO NACIONAL
ESCUELA SUPERIOR DE FÍSICA Y MATEMÁTICAS
Amigo 친구 global games

Questions

1) (a) Let x_1,x_2,x_3 be three distinct vectors in $(Z/3Z)^n$ such that $x_1=0$ and $x_2+x_3\neq 0$. Let y_1,y_2,y_3 be another set of three pairwise distinct vectors such that $y_1+y_2+y_3\neq 0$. Prove that there exists an affine transformation (that is, a linear transformation composed with a translation) A such that $A(x_i)=y_i$.
(b) Now consider the eight vectors in $(Z/3Z)^3$ given by: $x_1=(0,0,0)$ $x_2=(1,1,0)$ $x_3=(0,0,1)$ $x_4=(1,1,1)$ and $y_1=(0,0,0)$ $y_2=(1,0,0)$ $y_3=(0,1,0)$ $y_4=(0,0,1)$
Prove that there is no affine transformation mapping each x_i to the corresponding y_i.

2) Consider an m by n grid S of integers
$S_{1,1}$ $S_{1,2}$ ... $S_{1,n}$ . . . . $S_{m,1}$ $S_{m,2}$ ... $S_{m,n}$
We say that the {i,j}-th entry in this grid is "critical" if $S_{i,j}>3$.
If a grid has at least one critical entry, we say that this grid is "unstable", otherwise we call it "stable".
If the entry at {i,j} of S is critical, we define a "toppling" operation by forming a new grid S' with entries equal to entries of S, except S'*{i,j} whose value is obtained by subtracting 4 from S*{i,j}, and its up, down, left, and right neighbors, whose values are obtained by increasing their original values by 1. If a critical entry does not have all four neighbors, we still subtract 4 from its value and add 1 to each of the available neighbors. Thus S and S' will differ by at most 5 entries.
Show that any grid after a finite number of toppling applications converges to a unique stable grid, which we call its "stabilization". Show that this number does not depend on the choice of order of topplings. Write a program that takes as input a grid S of non-negative integers and 1) Outputs its stabilization S' and 2) Outputs the number of topplings required to arrive to S'.
Examples ("Input:" and "Output:" are headers and NOT part of the program's actual inputs or outputs)
Input:
2 2
2 4
Output:
3 3
3 0
1

Input:
3 5 4 2
2 3 4 8
1 1 1 1
Output:
2 2 1 2
1 0 2 3
2 3 3 3
13

3) Fix a positive integer n. By a net we mean a diagram consisting of cables and bars, such as Figure 1.
An admissible n-coloring of the net is a choice of permutation assignment to each bar (each bar acts as a connector that permutes strands passing through it) such that • The resulting diagram is a collection of n-disjoint loops (each loop represents a color). • No loop traverses the same bar more than once.
An admissible coloring is thus completely determined by the choice of permutations replacing the bars. For example, consider Figure 2, the left example has the identity permutation on each bar and it is an admissible 4-coloring. However, the right example is not an admissible 3-coloring, since the external loop passes through a bar more than once.
Exercise: in the net of Figure 3, • Find those admissible 6-colorings. • Find those admissible 5-colorings. (Each coloring in your answer must be presented as a picture.)

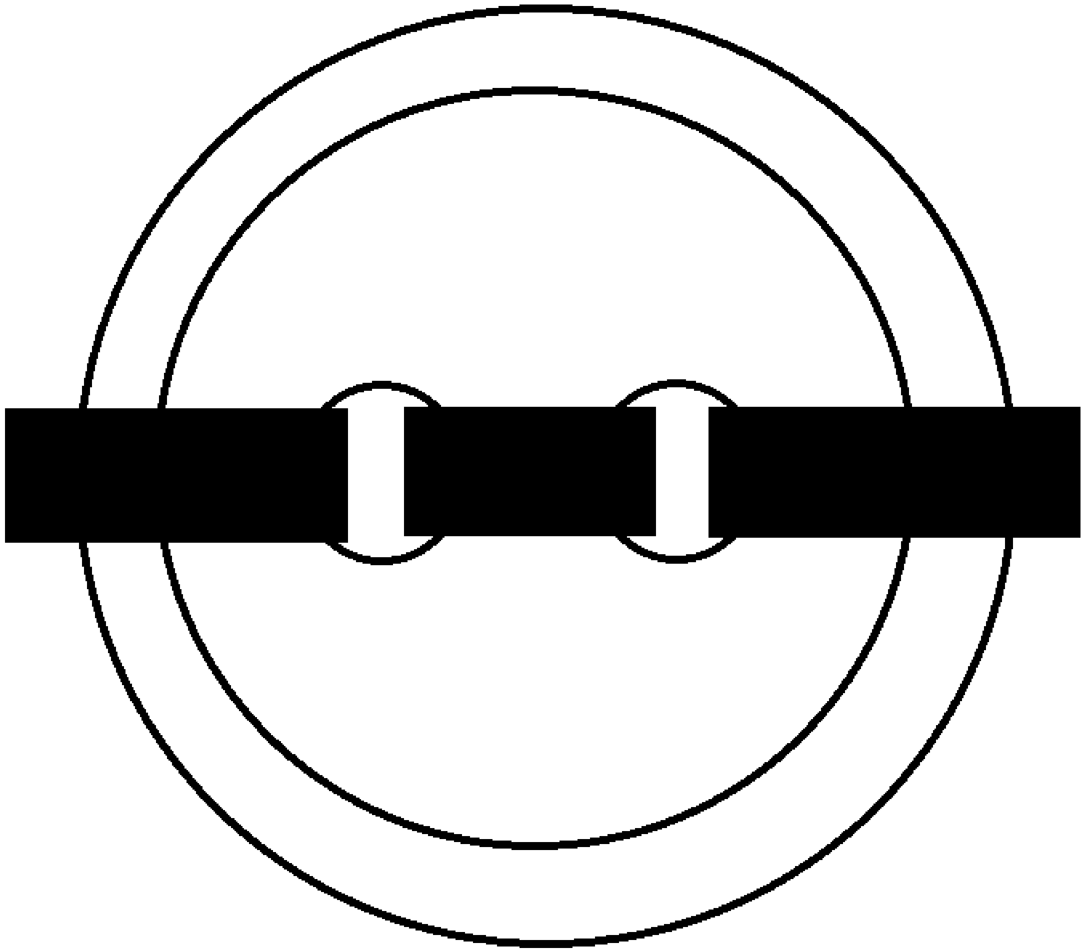

Figure 1

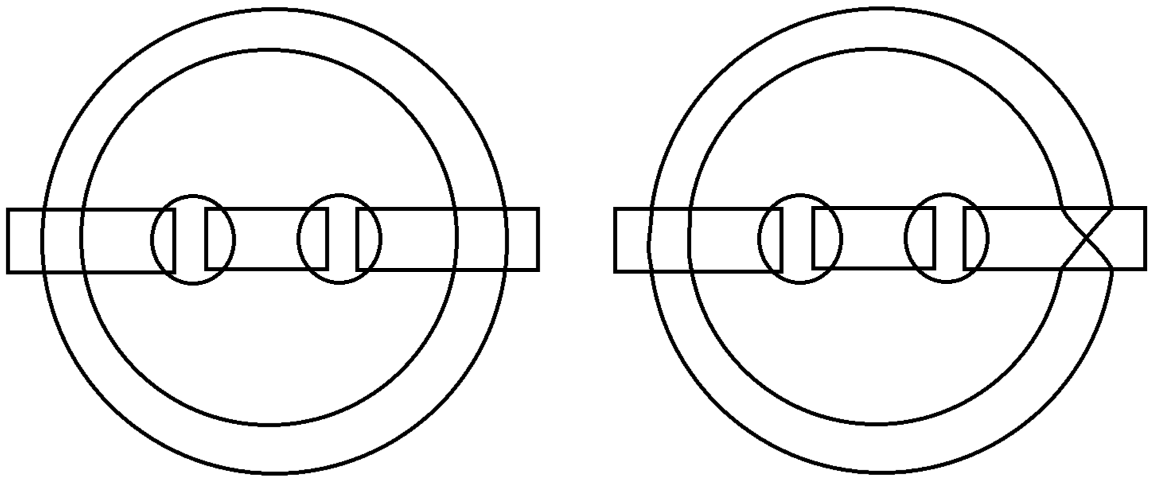

Figure 2

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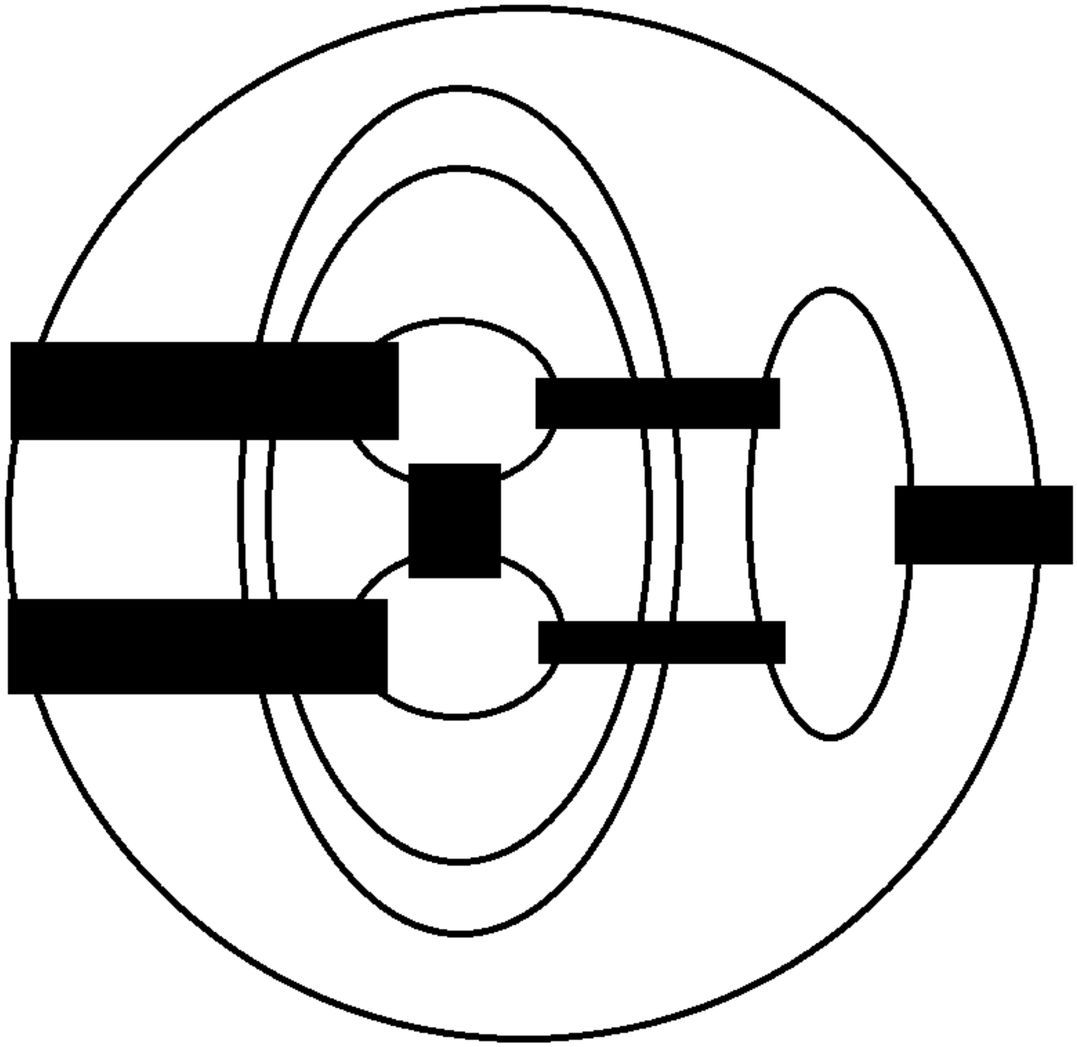

Figure 3